2 Superposition Principle and Coupled Oscillations

2.1 DEGREES OF FREEDOM

Number of independent coordinates required to specify the configuration of a system completely is known as degrees of freedom.

2.2 SUPERPOSITION PRINCIPLE

For a linear homogeneous differential equation, the sum of any two solutions is itself a solution.

Consider a linear homogeneous differential equation of degree n:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + ... + a_1 \frac{dy}{dt} + a_0 y = 0.$$

If y_1 and y_2 are two solutions of this equation then $y_1 + y_2$ is also a solution, which can be proved by direct substitution.

2.3 SUPERPOSITION PRINCIPLE FOR LINEAR INHOMOGENEOUS EQUATION

Consider a driven harmonic oscillator

$$m \frac{d^2 x}{dt^2} = -kx + F(t)$$

where $F(t)$ is the external force which is independent of x. Suppose that a driving force $F_1(t)$ produces an oscillation $x_1\,(t)$ and another driving force $F_2(t)$ produces an oscillation $x_2\,(t)$ [when $F_2(t)$ is the only driving force]. When the total driving force is $F_1(t) + F_2(t)$, the corresponding oscillation is given by $x(t) = x_1(t) + x_2(t)$.

2.4 SUPERPOSITION OF SIMPLE HARMONIC MOTIONS ALONG A STRAIGHT LINE

If a number of simple harmonic motions along the x-axis

$$x_i = a_i \sin(\omega_i t + \phi_i),\ i = 1, 2, .., N \qquad ...(2.1)$$

are superimposed on a particle simultaneously, the resultant motion is given by

$$X = \sum_i x_i = \sum_i a_i \sin(\omega_i t + \phi_i). \quad \text{...(2.2)}$$

2.5 SUPERPOSITION OF TWO SIMPLE HARMONIC MOTIONS AT RIGHT ANGLES TO EACH OTHER

If two simple harmonic motions

$$x = a \sin \omega_1 t, \quad \text{...(2.3)}$$

$$y = b \sin(\omega_2 t + \phi) \quad \text{...(2.4)}$$

act on a particle simultaneously perpendicular to each other the particle describes a path known as Lissajous figure when ω_1 and ω_2 are in simple ratio. The equation of the path is obtained by eliminating t from these two equations. The position of the particle in the xy plane is given by

$$\vec{r} = x\,\hat{i} + y\,\hat{j} \quad \text{...(2.5)}$$

SOLVED PROBLEMS

1. *Two simple harmonic motions of same angular frequency* ω

$$x_1 = a_1 \sin \omega t,$$

$$x_2 = a_2 \sin (\omega t + \phi)$$

act on a particle along the x-axis simultaneously. Find the resultant motion.

Solution

The resultant displacement is

$$X = x_1 + x_2 = \sin \omega t\, [a_1 + a_2 \cos \phi] + \cos \omega t\, [a_2 \sin \phi].$$

We put

$$R \cos \theta = a_1 + a_2 \cos \phi, \quad \text{...(2.6)}$$

$$R \sin\theta = a_2 \sin \phi \quad \text{...(2.7)}$$

so that

$$R^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos \phi \quad \text{...(2.8)}$$

and

$$\tan \theta = \frac{a_2 \sin \phi}{a_1 + a_2 \cos \phi} \quad \text{...(2.9)}$$

The resultant displacement is

$$X = R \sin(\omega t + \theta) \quad \text{...(2.10)}$$

which is also simple harmonic along the x-axis with the same angular frequency ω. The amplitude R and the phase angle θ of the resultant motion are given by Eqns. (2.8) and (2.9) respectively.

Special Cases

(*i*) $\phi = \pm 2n\pi$, $n = 0, 1, 2, ...$ or, the two SHMs x_1 and x_2 are in phase,

$$R = a_1 + a_2$$

(*ii*) $\phi = \pm (2n + 1)\pi$, $n = 0, 1, 2,$...or, the two SHMs x_1 and x_2 are in opposite phase,

$$R = a_1 \sim a_2.$$

In this case, the resultant amplitude is zero when $a_1 = a_2$ and one motion is destroyed by the other.

2. *Find the resultant motion due to superposition of a large number of simple harmonic motions of same amplitude and same frequency along the x-axis but differing progressively in phase.*

Solution

The simple harmonic motions are given by

$$\begin{aligned} x_1 &= a \sin \omega t, \\ x_2 &= a \sin(\omega t + \phi), \\ x_3 &= a \sin(\omega t + 2\phi), \\ &\vdots \quad \vdots \\ x_N &= a \sin[\omega t + (N-1)\phi]. \end{aligned}$$

The resultant displacement is

$$\begin{aligned} X = \sum_i x_i &= a \sin \omega t\ [1 + \cos \phi + \cos 2\phi + ...+ \sin (N-1)\phi], \\ &\quad + a \cos \omega t\ [0 + \sin \phi + \sin 2\phi + ...+ \sin (N-1)\phi], \\ &= R \sin (\omega t + \theta) \qquad ...(2.11) \end{aligned}$$

where $\quad R \cos \theta = a\ [1 + \cos \phi + \cos 2\phi + ... + \cos (N - 1)\phi],$
$\qquad\quad R \sin \theta = a\ [0 + \sin \phi + \sin 2\phi + ... + \sin (N - 1)\phi]$

Now, $\quad e^{i\phi} + e^{2i\phi} + ... + e^{i(N-1)\phi} = \dfrac{e^{i\phi}(e^{(N-1)\phi} - 1}{e^{i\phi} - 1}$

$$= e^{\frac{iN\phi}{2}} \frac{\sin (N-1)\phi/2}{\sin \phi/2}$$

Equating the real and imaginary parts, we get

$$\cos \phi + \cos 2\phi + ... + \cos (N-1)\phi = \frac{\cos N\phi/2 \sin (N-1)\phi/2}{\sin \phi/2}$$

$$\sin \phi + \sin 2\phi + ...+ \sin (N-1)\phi = \frac{\sin N\phi/2 \sin(N-1)\phi/2}{\sin \phi/2}$$

Thus, we write

$$\begin{aligned} 1 + \cos \phi + \cos 2\phi + ...+ \cos (N-1)\phi &= 1 + \frac{\cos N\phi/2 \sin(N-1)\phi/2}{\sin \phi/2} \\ &= \frac{\sin\{N-(N-1)\}\phi/2 \cos N\phi/2 \sin(N-1)\phi/2}{\sin \phi/2} \\ &= \frac{\sin N\phi/2 \cos (N-1)\phi/2}{\sin \phi/2} \end{aligned}$$

The resultant motion of Eqn. (2.11) is simple harmonic with amplitude and phase angle given by

$$R = a\frac{\sin(N\phi/2)}{\sin(\phi/2)} \quad ...(2.12)$$

$$\theta = (N-1)\phi/2 \quad ...(2.13)$$

When N is large and ϕ is small, we may write

$$\theta \approx N\phi/2, \quad ...(2.14)$$

$$R \approx Na\frac{\sin\theta}{\theta} \quad ...(2.15)$$

and the phase difference between the first component vibration x_1 and Nth component vibration x_N is nearly equal to 2θ.

The resultant amplitude may be obtained by the vector polygon method (Fig. 2.1). The polygon OABCD is drawn with each side of length a and making an angle ϕ with the neighbouring side. The resultant has the amplitude OD with the phase angle = $\angle DOA$ with respect to the first vibration.

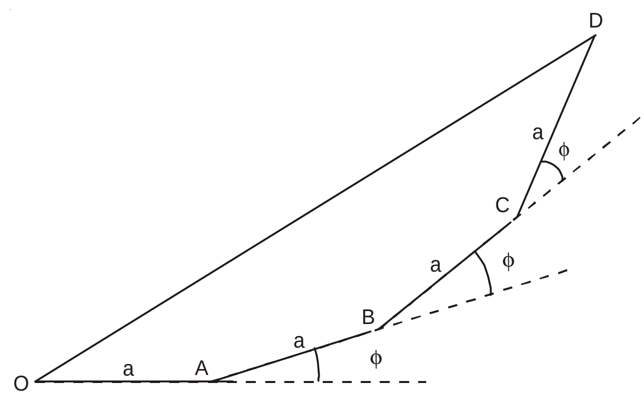

Fig. 2.1

Special Cases

(*i*) We consider the special case when there is superposition of a large number of vibrations x_i of very small amplitude a but continuously increasing phase. The polygon will then become an arc of a circle and the chord joining the first and the last points of the arc will represent the amplitude of the resultant vibration (Fig. 2.2). When the last component vibration is at A, the first and the last component vibration are in opposite phase and the amplitude of the resultant vibration = OA = diameter of the circle. When the last component vibration is at B, the first and the last component vibrations are in phase, the polygon becomes a complete circle and the amplitude of the resultant vibration is zero.

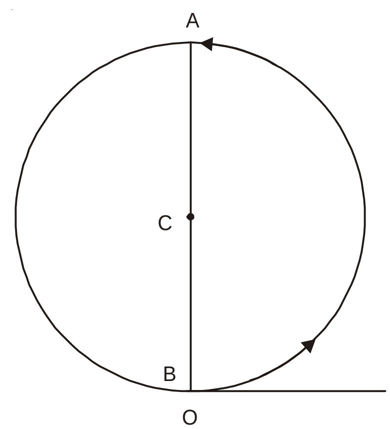

Fig. 2.2

(*ii*) When the successive amplitudes of a large number of component vibrations decrease slowly and the phase angles increase continuously the polygen becomes a spiral converging asymptotically to the centre of the first semicircle.